PETER S. HAMMERSTEIN

Stochastic Resonance and Noise-Assisted Signal Transfer

On Coupling-Effects of Stochastic Resonators and Spectral Optimization of Fluctuations in Random Network Switches

UPPSALA
UNIVERSITET

Dissertation for the Degree of Doctor of Philosophy presented at Uppsala University in 2004

Abstract

Hammerstein P. 2004: Stochastic Resonance and Noise-Assisted Signal Transfer. On Coupling-Effects of Stochastic Resonators and Spectral Optimization of Fluctuations in Random Network Switches. Acta Universitatis Upsaliensis. *Uppsala Dissertations from the Faculty of Science and Technology* 55. 117 pp. Uppsala. ISBN 91-554-5968-4.

Recent research shows that noise or random fluctuations must not always be destructive in Nature by degrading system performance. On the contrary, in nonlinear systems they can synchronize systems or enhance the quality of signal transmission. The latter possibility is reported in this thesis.

The phenomenon of *stochastic resonance* (SR) is presented and illustrated by an example of a ferromagnetically coupled spin chain, described by the Glauber's stochastic Ising spin model. It is demonstrated that an optimal strength of the next-neighbor interaction is able to improve the SR-effect. A similar mechanism has further been studied, both numerically and analytically, on the stochastic nonlinear dynamics of a ferromagnetic stripe domain in an inhomogeneous thin film. SR and its dependence on the domain stiffness, which is due to the exchange interaction, are presented. Experimental parameters for potential verification on Bi-doped epitaxial garnet-ferrite films are proposed.

Further-on, a nonlinear model of a junction in neuronal and road structures is studied using various types of noise (stochastic processes) to generate the incoming traffic. It is shown that random fluctuations are able to enhance signal transmission, whereby the zero crossings of colored ($1/f^k$) Gaussian noise is superior to Poissonian noise and, in certain cases, to deterministic, periodic traffic too. Optimal traffic for $k \approx 1$ has been found. In case of Gaussian $1/f^k$ noise modulated periodic input, noise-assisted traffic can be observed as well and demonstrate how random fluctuations can enhance the signal traffic efficiency in a network. The effect of an optimal k has finally been applied to a data package network switch, whereby a stochastic data scheduling algorithm is proposed and investigated numerically and analytically.

Peter Hammerstein, Department of Engineering Sciences, The Ångström Laboratory, Uppsala University, Box 534, SE-751 21, Uppsala, Sweden.

ISSN 1104-2516 ISBN 91-554-5968-4
Printed in Sweden by Elanders Gotab, Stockholm 2004
Distributor: Uppsala University Library, Box 510, SE-751 20 Uppsala, Sweden.
www.uu.se, acta@ub.uu.se

God not only play dice, but
sometimes throws them,
where they cannot be seen.
(Stephen Hawking)

List of Publications

Licentiate Thesis:

The Benefits of Noise: Stochastic Resonance and Noise Assisted Signal Transfer
Peter S. Hammerstein
Licentiate Thesis, Uppsala University (2002).

Journal papers:

I. Stochastic Resonance in Ferromagnetic Domain Motion
P.S. Ruszczynski[1], L. Schimansky-Geier, and I. Dikshtein,
Eur. Phys. J. B **14**, 569 (2000).

II. Noise Enhanced Efficiency of Ordered Traffic
P.S. Ruszczynski[1] and L.B. Kish,
Phys. Lett. A **267**, 187 (2000).

III. Noise-Assisted Traffic of Spikes through Neuronal Junctions
P.S. Ruszczynski[1], L.B. Kish, and S. Bezrukov,
Chaos **11**, 581 (2001).

IV. Spectral Optimization of Computer Network Traffic in a Stochastic Data Packet Scheduling Algorithm Triggered by $1/f^k$ Noise
P. Hammerstein and A. Ramanujam,
submitted , (2003).

[1] now: Hammerstein

Articles in conference proceedings:

V. Symbolic Representation of Neuronal State Space Dynamics
K. Stiefel , P. St. Ruszczynski[2] and R. Lakatos,
in *Student Papers of the Complex Systems Summer School 2001*, (Santa Fe Institute, U.S.A., 2001).

VI. Fluctuations of Cars and Neural Spikes at Junctions
P.S. Ruszczynski[2], L.B. Kish, and S. Bezrukov,
in *Proceedings of the 16th International Conference on Noise in Physical Systems and 1/f Fluctuations ICNF 2001*, Gainesville, Florida, U.S.A., edited by G. Bosman, (World Scientific, Singapore, 2001).

[2]now: Hammerstein

Contents

1 INTRODUCTION **1**

1.1 Complex Systems Theory 2

1.1.1 Continuous cases 2

1.1.2 Discrete cases 10

1.2 Stochastic Processes 11

1.2.1 Mathematics of stochastic processes 11

1.2.1.1 Processes without memory 12

1.2.2 Stochastic processes in physics 15

1.2.3 The power spectral density of noise 16

1.2.3.1 $1/f^k$ noise 18

2 STOCHASTIC RESONANCE **21**

2.1 Dynamical Stochastic Resonance 23

2.2 Non-Dynamical Stochastic Resonance 29

2.3 Generalizations and Examples 33

2.3.1 Quantifiers of stochastic resonance 33

2.3.2 Related phenomena 34

2.3.3 Stochastic resonance and the solid state . . 35

2.4 Stochastic Resonance in Ferromagnetic Domain Motion 40

2.4.1 Lagrangian equation of motion 43

2.4.2 Stationary case without temporal forces . . 46

2.4.3 Numerical treatment 48

2.4.4 Theoretical approach 52

3 RANDOM NETWORK SWITCHES **57**

3.1 A Traffic Model of Neuronal and Road Networks . . 57

3.1.1 Noise triggered input 60

3.1.1.1 Poissonian process 60

3.1.1.2 Colored noise 62
3.1.2 Strictly periodic input 66
3.1.3 Noise modulated periodic input 68
3.1.3.1 Gaussian white noise 68
3.1.3.2 Gaussian colored noise 69
3.2 A Stochastic Data Packet Scheduling Algorithm . . 71
3.2.1 Scheduling algorithms 72
3.2.2 A stochastic scheduling algorithm 73
3.2.3 Performance measure 75
3.2.4 Simulations and discussion 76

4 SUMMARY & OUTLOOK 83

ACKNOWLEDGMENTS 87

BIBLIOGRAPHY 89

APPENDIX 103

Chapter 1

INTRODUCTION

Most of the processes occurring in Nature share two properties: they are *nonlinear* and they are affected by *stochastic noise*. Scientists usually seek to neglect those facts because the theoretical modeling and description can get rather complicated. A linear, deterministic theory is preferred instead and often sufficient. But a lot of features and phenomena can not be captured by that.

Due to the enormous increase of computational power and the development of new physical fields, the treatment of stochastic, nonlinear systems became easier to handle and, hence, very popular. Special attention has been paid to effects where the stochastic noise, or fluctuations, do *not* degrade the performance of a system as it is often the case, but instead provides a useful and necessary tool to perform signal detection, enhance signal transmission, synchronize systems, form patterns and structures, etc. This can be the case if the system has nonlinear characteristics.

Nonlinearity means that the underlying dynamic (differential) equations are nonlinear in the independent variable, i.e., their typical solutions can not be expressed as a linear combination of elementary solutions. The branch of physics studying those systems is called *Complex Systems*, *Nonlinear Dynamics* or *Dynamical Systems*. The next section provides a brief overview over this branch.

Stochasticity means that random fluctuations can occur biasing the system in a probabilistic manner. Noise affects all kind of natural systems, often deteriorating the predictability of the future system state. The study of noise in physical, chemical and biological systems has been performed in branches like *Non-equilibrium*

Statistical Physics (Mechanics) and *(Applied) Stochastic Processes*.

The present treatise considers stochastic effects in nonlinear systems as a model of noise in physical and other systems.

In the following, short introductions are given to both fields, Complex Systems Theory and Stochastic Processes. Since the presented work focuses mainly on details of stochastic processes, the reader may consider the next Sec. 1.1 as an interesting trip into a modern discipline of physics, describing the more profound framework the presented phenomena should be seen in context to.

1.1 Complex Systems Theory

The study of complex systems (which not necessarily have to be complicated) can mainly be divided in

1. the analysis of problems continuous in time and/or space, i.e.,
   - nonlinear (partial) differential equations and
2. the analysis of problems discrete in time and/or space, e.g.,
   - discrete mappings, (complex[1]) number iterations
   - cellular automata.

Complex hereby does not necessarily mean that the systems have a very high number of degrees of freedom. Complex is rather meant as a distinction from simple systems which can sufficiently be described by linear mathematics. An overview on literature (journals, conference proceedings, textbooks and important papers) on this topic can be found in a Resource Letter at [1].

1.1.1 Continuous cases

The study of nonlinear differential equations arouse more than a century ago with problems of oscillations in classical mechanics and electric circuits (Duffing oscillator [2], van der Pol's equation [3]).

[1] Here, complex is meant in the mathematical sense as a linear combination of real and imaginary numbers, in contrast to *"complex"* in the physical terminology of complex systems.

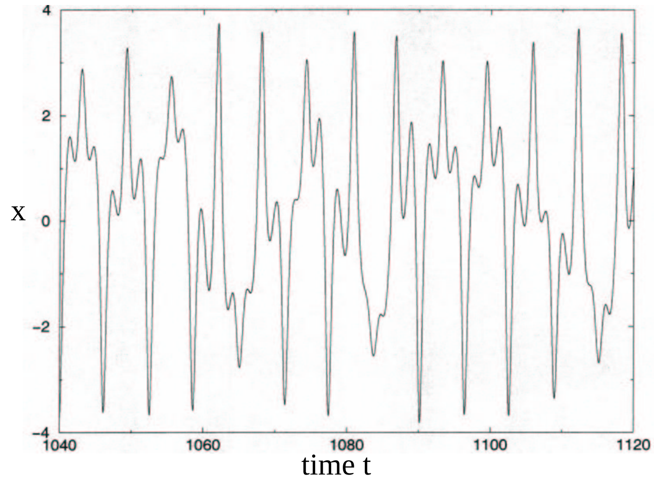

Figure 1.1: Chaotic oscillation: Simulation of the location X as a function of time of the nonlinear Duffing oscillator.

Driven and damped oscillation equations in various nonlinear potentials exhibited qualitatively new kinds of solutions, such as a dependence between amplitude and frequency of the oscillations. The analysis of the evolution of the trajectories in the state (phase) space discovered a topology which is more complex and exhibits new qualitatively distinct features than in simpler linear problems. An example for chaotic oscillation is given in Fig. 1.1.

Since a differential equation of n-th order can always be expressed as a system of n coupled differential equations of 1st order, n is called the dimension of the state space.

Coupled ordinary linear differential equations already exhibit a number of interesting behavior of the movement of the trajectories, such as stable and unstable fix-points (nodes) and stable and unstable foci, where the trajectory will be attracted (spiral in towards the fixpoint) or repelled (spiral away from the fixpoint).

Others, so called saddles (hyperbolic points) attract/repel trajectories depending on their initial condition. In that case there is always one trajectory separating those regions and therefore called the separatrix. There may exist other singular points too, so called centers (elliptic points), where the trajectories follow closed ellipses around them. Those points are neither attractive nor repulsive. Which of the above mentioned behaviors eventuate depends on the dimensions and parameters of the system determining the signs and values of the (complex) eigenvalues.

Since the stability analysis of nonlinear solutions can (often) be carried out by a linearization around the singular points, results for the linear problems can be of use for the general nonlinear case as well.

However, in the case of nonlinearities one can observe phase transitions of the first and second kind in only one dimensional problems, e.g., by studying a single nonlinear differential equation of first order. An example is the kinetic description of an autocatalytic reaction. In two dimensions *bifurcations* can occur. Bifurcations are qualitative changes of the topology of the state space, caused by parameter variation (see Fig. 1.3). For example, a stable focus can become unstable (Hopf-bifurcation).

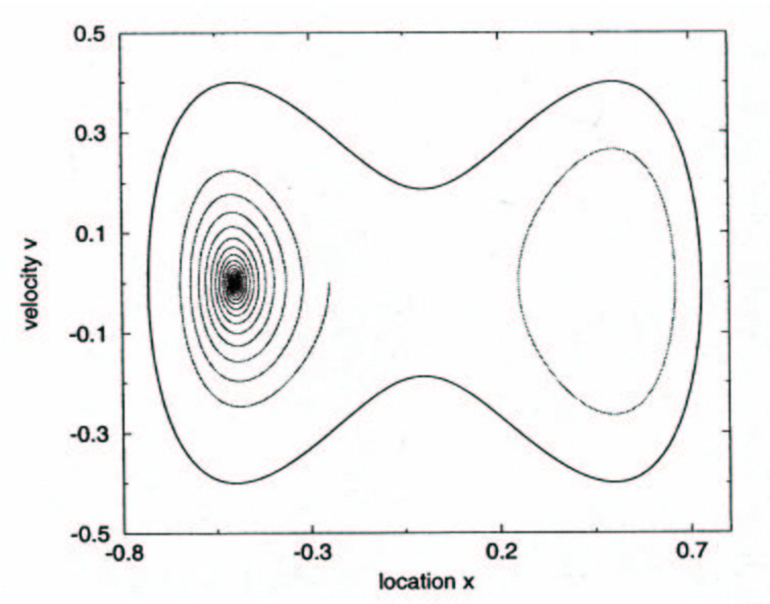

Figure 1.2: Different kinds of solutions in phase space of the nonlinear Duffing oscillator as a result of different initial conditions and parameters. The inner closed curve on the right hand side is the analogy to the undamped harmonic oscillator, whereas the spiral on the left hand side corresponds to the damped case. The enveloping curve is a characteristic example of a nonlinear oscillation.

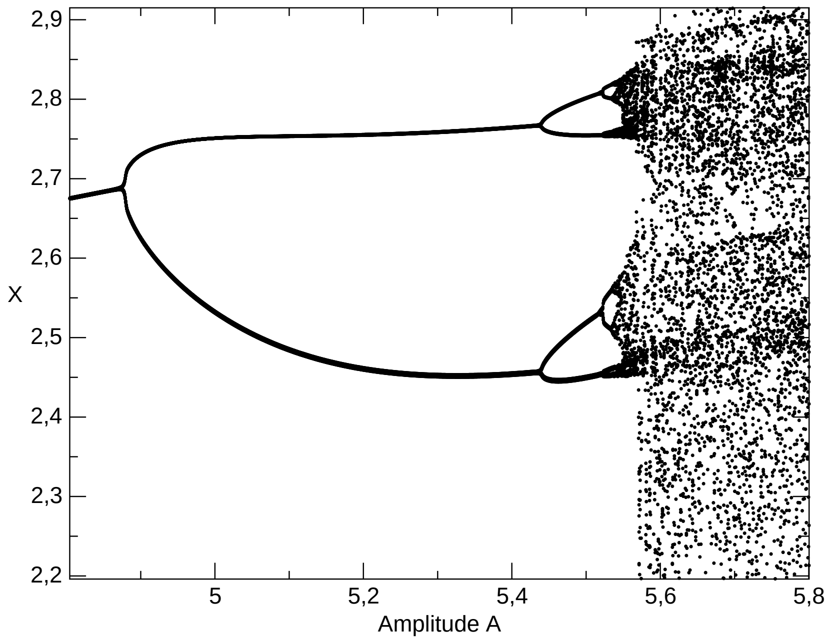

Figure 1.3: Bifurcation diagram of the nonlinear Duffing's oscillator. As a control parameter A is varied, the topology of the state space is changed, leading to qualitatively different types of solutions. The example shows the well studied period-doubling route to chaotic behavior which occurs at a parameter value of about A = 5.57.

Considering three dimensions, quasiperiodic solutions called tori can determine the ultimate destiny of the trajectories and a phenomena known as *chaos* can occur.

In general, chaos is possible in autonomous differential equations if at least three coupled equations are present containing at least one nonlinearity. In case of non-autonomous system already two (!) coupled equations are sufficient to observe chaotic behavior. This emphasizes the importance of the systematic study of complex systems and their specific solutions.

There are several definitions of chaos all having one feature in common: the sensitivity of the systems development to a slight change in the initial conditions, which can be measured by the Ljapunov exponent.

However, chaos does not mean disorder in a stochastic sense. The movement of the trajectory is absolute deterministic, although the plot of two trajectories starting nearby will reveal an exponential growth of their distance (at least for a while), demonstrating the sensitivity to the initial conditions (see Fig. 1.4). But, in many cases the trajectory will never leave a bounded region and end up in a so called strange attractor (Figs. 1.4 and 1.5).

Although the path of a trajectory might be difficult to predict, the shape of the strange attractor will always be the same, e.g., showing the same *fractal* (self similar) properties as shown in Figs. 1.5 - 1.7.

In parameter space different routes to chaos have been found. One of the most interesting one is the so called period doubling which is displayed in Fig. 1.3. At certain parameter values the system oscillates with a fixed period while changing the parameter suddenly leads to a doubling of this period. This can happen

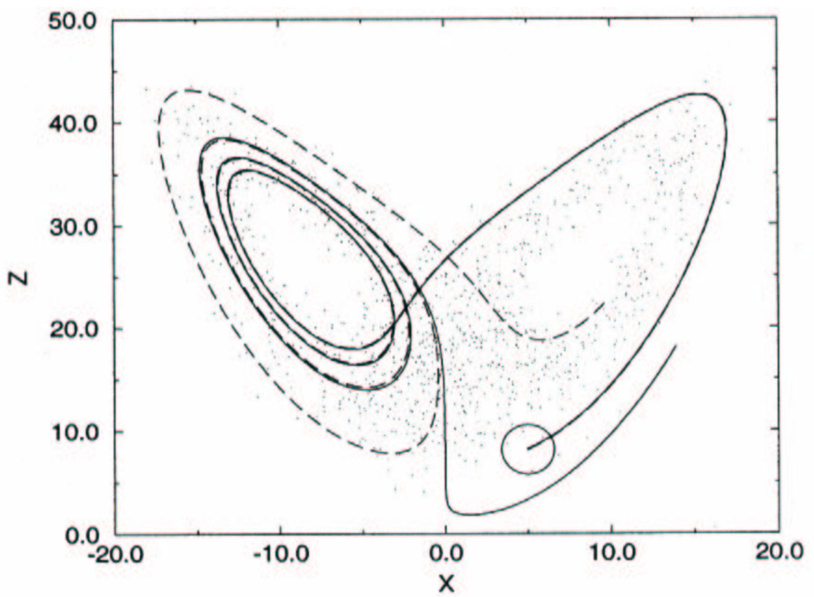

Figure 1.4: The sensitivity of the path of a trajectory in state space to its initial condition is what is called *chaos* in physics. Two trajectories starting very close to each other (her indistinguishable in the small circle), will separate exponentially from each other and finally end up in different regions of the state space. The example, provided here, is taken from the so called strange attractor of the meteorological Lorenz model and suggests the shape of a butterfly. See also Fig. 1.5.

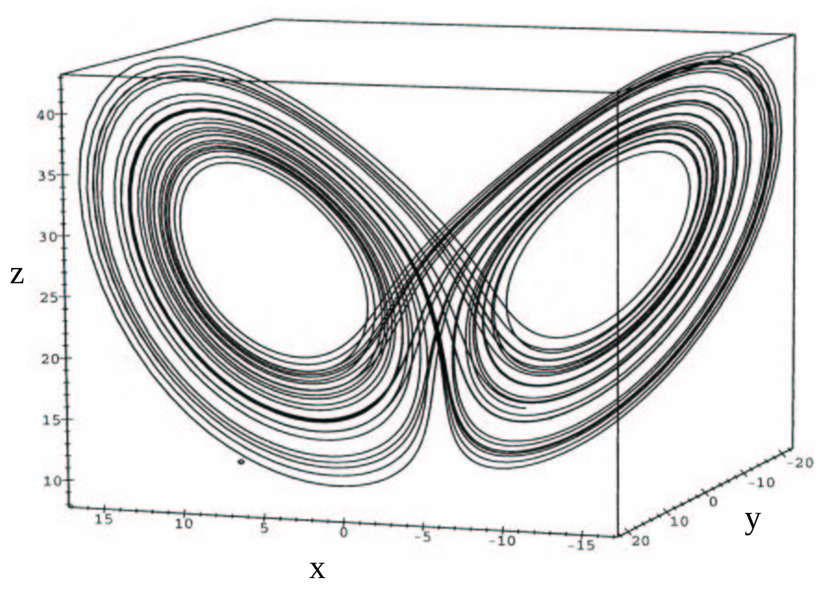

Figure 1.5: The Lorenz attractor as an example of a so called strange attractor. See also Fig. 1.4

when a limit cycle becomes unstable while at the same time two stable limit cycles arise (bifurcation). A further changing of the bifurcation parameter doubles the period again, and so on, until the system ends up in a chaotic region. The ratio of two consecutive parameter values where the period doubles has been found to be a universal scaling constant [4], [5]. If the above mentioned instability continuous as a branch in the parameter space and hits a chaotic region, one dramatically calls this an explosion of chaos or a *crisis*.

Examples of the mentioned (and to be mentioned) mechanisms and phenomena can be found in every field of every scientist.

The study of population dynamics in ecology, i.e., the coupled processes of growth and decay of different species concentrations, has played a key role in the development of the field of complex systems. The spread of epidemics, evolutionary processes, solar systems dynamics or model deduction based on the analysis of time series (such as financial data or electrocardiogram sequences) are further examples.

One important consequence of nonlinearity in coupled differential equations is a principle studied by Hermann Haken in laser physics; it has been called *slaving principle*. It requires a set of equations (variables) which evolve within different characteristic time scales in a way that the fast varying functions can be eliminated adiabatically. The result is that the time course of one variable (master mode) determines the evolution of the others (slave modes). Thus one can say that the system is "organizing" itself into a certain mode which for this reason is called *self-organization* or *synergetics* [6], [7].

But nonlinear effects are of course not restricted to temporal phenomena only. Considering the well studied reaction-diffusion equation one can observe spatial pattern formations. All what is needed for this observation is a diffusion besides nonlinear terms. A hydrodynamical example is the coupling between convective motion and thermal conduction in a liquid heated from below. At a certain value of a system parameter a hexagonal structure will appear on the surface, indicating a regular cellular structure of the

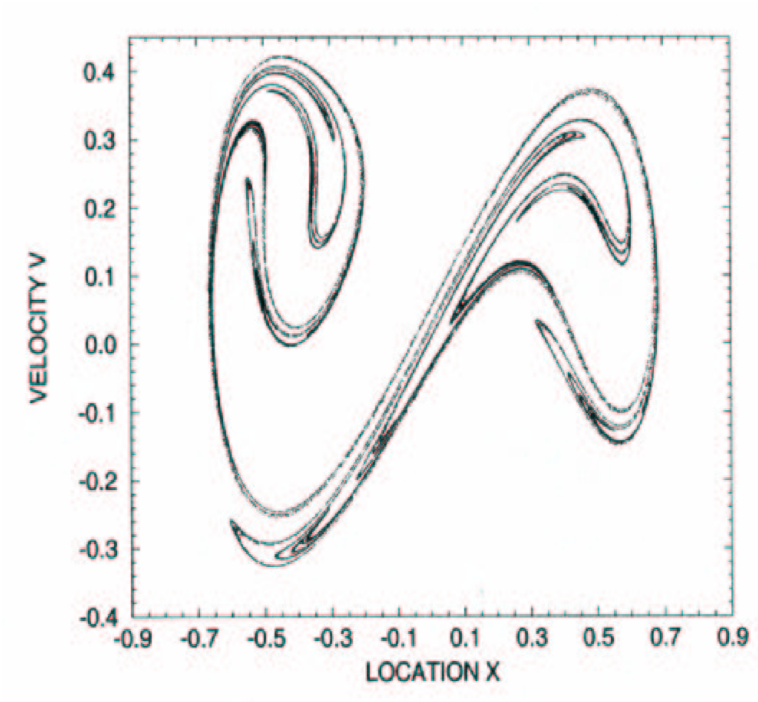

Figure 1.6: A typical example of the properties of a strange attractor. Here, the fractal Poincaré section of the strange attractor of the Duffing oscillator.

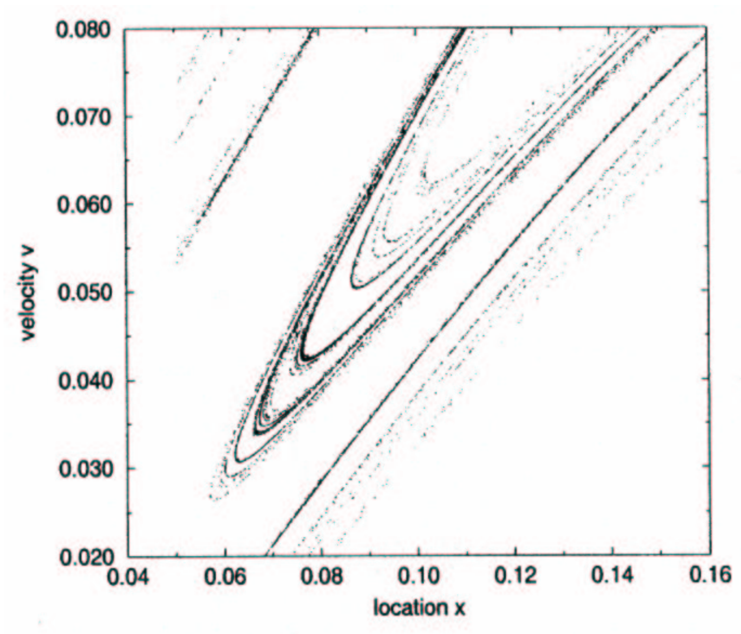

Figure 1.7: Enlargement of the Poincaré section of the strange attractor from Fig. 1.6 shows fractal properties.

heat transportation within the liquid (Rayleigh-Bénard instability [8], Lorenz model [9], Fig. 1.5).

Other examples can be found in plasma physics or chemical reactions such as the Zhabotinsky-Belusov reaction. Here, concentrations in a two dimensional layer vary with space and time, forming wave patterns for example as spirals [10]. Nonlinear waves are very interesting solutions of nonlinear partial differential equations. They do, for example, not show interference, i.e., the superposition principle is not valid here. Nonlinear waves called *solitons* are stable to perturbations and can interact like particles, i.e., with conservation of momenta and energy. Therefore a soliton can travel infinitely long without losing the particular shape.

One of the most simple nonlinear dynamics is given by the free motion of a particle in a bistable double-well potential, providing two stable solutions separated by an unstable one. Depending on the initial condition the particle will come to rest at either of the minima. Contrary, when the particle is forced by deterministic and stochastic forces, unexpected phenomena like dynamic stochastic resonance can occur, as it is introduced in this thesis in Chapt. 2.1 and 2.4.

1.1.2 Discrete cases

Often the complex behavior of a system can most easily be studied by discretization in space or time. Using the Newton-Raphson method, for example, solutions of nonlinear equations can be found. However, spatially and temporarily discrete problems also arise naturally such as the coupled behavior of single elements often studied as *cellular automata* or the discrete time analysis of iterative expressions such as the logistic equation describing populations of successive generations. The latter one has become famous since despite the simple mathematical formulation a very rich behavior could be found, verifying experimental data such as the periodic variations in the catch reports of the Hudson Bay Company from 1850 to 1930 [11]. Feigenbaum has studied the abstract formulation in great detail discovering periodic and chaotic behavior in the system depending on the parameters. While the parameter is changed, the qualitative behavior changes leading to the Feigenbaum scenario of bifurcations. The universal scaling law, mentioned above, has been observed for the constant ratio of consecutive bifurcation parameter values [4], [5]. Many other maps have been studied such as the Poincare map or the Henon map [12] showing fractal properties in the corresponding attractors.

The fact that already one dimensional problems can give rise to chaotic solutions in case of discrete nonlinear systems underlines the importance of the detailed study of this field.

Simple rules can supply a possibility to create patterns that seem rather complicated. Well known are the fractals of the Cantor-set, the Koch-curve and the Sierpinsky-gasket [13]. Cellular automata in different dimensions show similar behavior. A cellular automata can be understood as a set of elements on a grid whereby the time evolution of the elements depend on the state of the neighbors. Very simple update rules for the next time step can lead to surprisingly complex structures as shown in Fig. 1.8. Examples and applications for the studied models can be found in all kinds of networks.

As mentioned above, a discrete description can arise from the discrete nature of the studied objects. Examples modeled and stud-

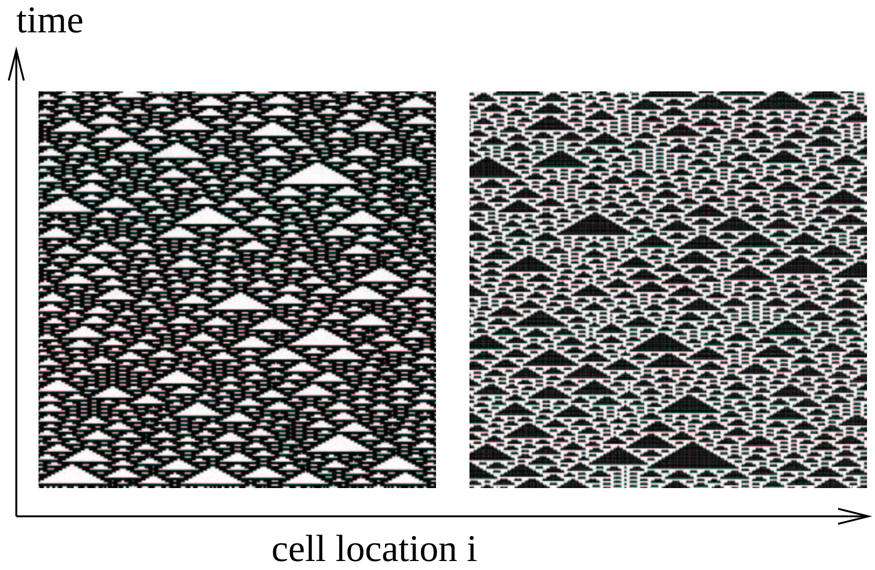

Figure 1.8: An example for a discrete complex system: Spatio-temporal structure formation in a one-dimensional cellular automaton. Very simple interaction rules between adjacent cells can lead to complex and fractal structures.

ied in Chapt. 3.1 of the present thesis are traffic systems of cars and neural spikes which are obviously discrete in their nature.

1.2 Stochastic Processes

1.2.1 Mathematics of stochastic processes

A stochastic process can be defined as a process $Y(X,t)$ depending on a random number X. This is the time dependent case of a more general definition of a random function. As in the previous sections the time dependence can be of discrete or continuous nature. The capital letters X and Y stand for random variables, i.e., an ensemble of their concrete realizations which shall be denoted by x and y. The probability that the realization y eventually will occur is described by a probability distribution function $P(y)$. Often the term probability density $p(y)$ is used. Then $\int_{y_1}^{y_2} p(y)dy$ gives the probability to find a value y within the interval (y_1, y_2) in the continuous case.

A discrete process Y develops stepwise $(..., y_{n-1}, y_n, y_{n+1}, ...)$ and is described by the probability $P(y_n)$ that y_n appears at time t_n. The dependence of y_n on the previous values of Y is specified by the conditional probability $P(y_n|y_{n-1}, y_{n-2}, y_{n-3}, ..)$, which is the probability that y_n appears at time t_n supposed that the realization of Y at time t_{n-1} was y_{n-1}, at time t_{n-2} was y_{n-2} and so on. Defining the joint probability $P(y_n, y_{n-1})$ as the probability that y_n appears at time t_n and y_{n-1} at time t_{n-1} one can write down the basic expression

$$P(y_n, y_{n-1}) = P(y_n|y_{n-1})P(y_{n-1}) \quad , \tag{1.1}$$

known as Bayes' rule. In case that y_n does not depend on y_{n-1}, i.e., $P(y_n|y_{n-1}) = P(y_n)$, it follows that

$$P(y_n, y_{n-1}) = P(y_n)P(y_{n-1}) \quad , \tag{1.2}$$

which is the most fundamental law in probability theory and means statistical independence of the events y_n and y_{n-1}.

1.2.1.1 Processes without memory

If the state y_n of a process does not depend on the entire past, i.e., only on a finite number k of previous steps this process is called a *Markov process* and the conditional probability reduces according to

$$P(y_n|y_{n-1}, y_{n-2}, y_{n-3}, .., y_{n-\infty}) = P(y_n|y_{n-1}, y_{n-2}, y_{n-3}, .., y_{n-k}). \tag{1.3}$$

Note that this is a general definition and defines a Markov process of kth order. Most common in the literature is a definition which considers only the previous time step, i.e., $k = 1$. This leads to

$$P(y_n|y_{n-1}, y_{n-2}, y_{n-3}, .., y_{n-\infty}) = P(y_n|y_{n-1}) \quad . \tag{1.4}$$

This definition is indeed closer to Markov's original reflections from 1911 [14]. The process is entirely determined by the transition probability $P(y_n|y_{n-1})$ and can successively be constructed. Except for the knowledge of the last step this process has no memory. This is the Markov property and describes the class of stochastic processes which is the most important one in Nature:

Markov Process Examples

- *Radioactive Decay:* The stochastic number of nuclei changes according to the transition probability $P(n, M, t)$ where M and n are the number of pre-reaction nuclei at time 0 and time t respectively. This discrete process does obviously depend on the number of present nuclei only and not on the previous past.

- *Chemical Reactions:* The situation for simple chemical reactions involving the transition between two states is similar to the radio-active decay. Again the transition rate is proportional to the number of pre-reaction atoms or molecules, respectively.

- *Spin Relaxation Model:* Considering a two state system for a single spin, i.e., two possibilities (+ and -) for the directions of a spin one can write down the stationary transition probability and describe a system of spins as it relaxes to the equilibrium.

- *Random Walk:* This is a discrete model useful to describe Brownian motion. Here the direction at each step does not depend on the preceding steps. Brownian motion itself is the most important example of a Markov process in physics.

- *Poisson Process:* This is a point process with independent events on a real (time) axis. The possibilities of application in physics span over a wide range: the counts in a Geiger counter, the arrivals at the anode of a vacuum tube or the energies of cosmic ray particles [15]. Other examples are learning processes in neural networks [16] or stochastic resonance in neuron models [17]. The Poisson process is a special case of a generation-recombination process having a range of integers n which are occupied by the probability p_n. The time evolution of probability density functions is described by so called Master equations. In this case by

$$\dot{p_n} = \nu(p_{n-1} - p_n) \quad . \tag{1.5}$$

Thus the time dependent probability density for the Poisson process is given by

$$p_n(t) = \frac{(\nu t)^n}{n!} \exp(-\nu t) \quad . \tag{1.6}$$

It can be shown that ν is the mean value (rate) of the time gap between successive Poisson events as well as the variance.

As the reader will see later in this treatise, the Poisson process with varying mean rates ν will play a major role in the investigation of a stochastic traffic model.

To describe the time evolution of a stochastic process, two different, but mathematically equivalent, formalisms are common in use. The first is the Fokker-Planck equation [18], which is based on the more general Master equation. The second formalism is based on the Langevin equation. While the Fokker-Planck equation is a partial differential equation for the evolution of the probability density distribution, the Langevin equation is a differential equation for the random variables. Depending on the physical situation, the stochastic variables can enter the equation in additive or multiplicative terms. In this treatise we will see an example of either possibility.

As an example for a the Lagrange equation one can take a look at the equation of motion for a Brownian particle at position x

$$m\ddot{x} = -\alpha\dot{x} + \xi(t) \quad , \tag{1.7}$$

with mass m, friction constant α and the random force $\xi(t)$ having a mean value $< \xi(t) >= 0$.

The two mentioned formalisms are equivalent and suitable for linear problems but have to be handled carefully in nonlinear situations. A major problem is the integration of a stochastic (partial) differential equation since the added stochastic process $\xi(t)$ enters the equations as a random number sequence $y(t = n\epsilon), n \in \mathbb{N}$ of discrete nature [2]. While integrating over a small time interval $(t, t+\epsilon)$, the question arises which value of y should be chosen. There are two main approaches to the problem proposed by Itô $(y(t))$ and Stratonovic $(\frac{1}{2}(y(t) + y(t+\epsilon)))$. For an analysis of these problems see [15].

[2]The discrete character is a consequence of the Kramers-Moyal expansion [15] of the corresponding Master equation.

1.2.2 Stochastic processes in physics

Our most fundamental approach to describe Nature, the Quantum-theory, is a probabilistic one containing unpredictability at its deepest level. The stochasticity of the radio-active decay, for example, is a direct consequence of the quantum-mechanical formalism. Although probabilities are the primary quantities which can be determined in Quantum mechanics, the macroscopic laws always appear after integration (averaging) in Hilbert space.

This can be seen in analogy to the ensemble average in systems with many degrees of freedom in classical mechanics, which creates the basis for Statistical Physics[3]. Here one is able to deal with high-dimensional systems with complicated inherent dependencies.

The most famous example is provided by the Brownian motion. The force exerted by a very large number of molecules, acting on a large particle is changing very fast and is practically impossible to calculate using Newton's equation of motion. On the other hand, it is possible to average over small time intervals and reveal the macroscopic properties of the system, such as the validity of the damping law for the average velocity.

This is in fact the basic procedure. Considering different time scales, one can average out the fast varying variables and obtain equations for the remaining slow ones which establishes known macroscopic laws, such as Ohm's law or heat conduction. The interesting feature of Nature is that those laws are described by smooth functions, although they are based on the irregular microscopic motion.

But it is clear that the macroscopic laws do not describe the whole truth since they neglect the intrinsic fluctuations which appear as *noise* in many physical and biological systems [15].

In linear systems which are in equilibrium the random forces act as fluctuations around a certain mean value, only. This has been studied in equilibrium statistical mechanics and is well understood. In contrast, the study of non-equilibrium statistical mechanics is

[3] The average of a high-dimensional system can be the average over an ensemble (many different realizations of the same physical system) or the time average of one of the realizations in case that they are the same. If so, one calls the system ergodic. Ergodicity is the main assumption for equilibrium statistical mechanics.

relatively underdeveloped. A way to describe those systems is possible through the investigation of nonlinear stochastic dynamics, which the central theme of this thesis. Identifying and understanding nonlinear stochastic mechanisms and phenomena is a fruitful challenge and promising source of knowledge for all kind of scientific fields, especially solid state physics.

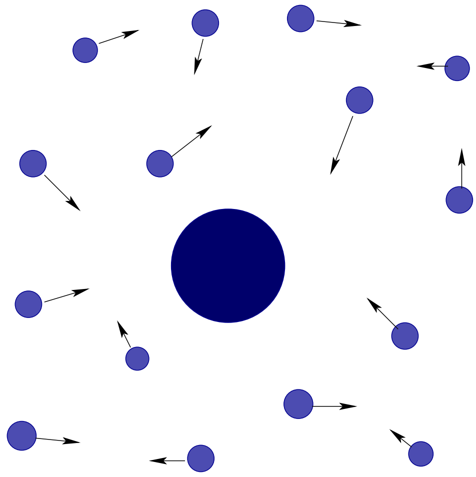

Figure 1.9: Brownian motion in two dimensions: fast varying, irregular, deterministic, microscopic forces acting "as random" on a mesoscopic particle.

1.2.3 The power spectral density of noise

As introduced in subsection 1.2, stochastic processes are often characterized by their probability distribution functions $P(y)$. However, the distribution of energy to the different frequencies $\omega = 2\pi f$ plays a very important role in many physical processes and is described by the power spectral density (PSD) $S(\omega)$. As with any time dependent deterministic function, the Fourier transform can formally be defined for a stochastic process too. Moreover it can be shown quite easily, that the PSD for a stationary process is defined by the Fourier transform $\mathfrak{F}()$ of the autocorrelation function

$$C_{y(t)\,y(t+\tau)}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\,y(t+\tau)\,\mathrm{d}t \quad . \tag{1.8}$$

Using the relation between the PSD and the Fourier transform

$$S(\omega) \;=\; \lim_{T\to\infty} \frac{1}{2T} |\mathfrak{F}(C(\tau))|^2 \quad , \tag{1.9}$$

one can write

$$\begin{aligned} S(\omega) &= \mathfrak{F}(C(\tau)) \qquad (1.10)\\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\tau)\, \exp(i\omega t) \quad . \end{aligned}$$

This is the well-known Wiener-Khintchine Theorem [19], [20].

Returning to the example of Brownian motion one can note that the value of the force, acting on the Brownian particle is independent on the position and velocity. Moreover the value of the force itself is markovian, i.e., does not depend on earlier values. Thus the autocorrelation function is a delta function:

$$\begin{aligned} C_{\xi(t)\,\xi(t+\tau)}(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(t)\, \xi(t+\tau)\, \mathrm{d}t \qquad (1.11)\\ &= 2\alpha k T\, \delta(\tau) \quad . \end{aligned}$$

The proportionality factor enters for consistency reasons with Eq. (1.7) and the equipartition law

$$< m\frac{\dot{x}^2}{2} >= \frac{3}{2} kT \quad . \tag{1.12}$$

Applying the Wiener-Khintchine theorem (1.10), it is now easy to calculate the power spectral density of the force as

$$S_\xi(\omega) = \frac{\alpha k T}{\pi} \quad . \tag{1.13}$$

Note that $S_\xi(\omega)$ is a constant and does *not* depend on the frequency ω. This means that the energy of the process is uniformly distributed to all frequencies or colors (in analogy to light). Therefore this is called *white noise*. Contrary to this mathematical result, it is clear that there are physical arguments for a cut-off frequency of the uniform spectrum. Otherwise the process would contain an infinite amount of energy.

The result for the white noise is based on the fact that there is no memory in the process and the correlation function of the random force is a delta function. However, this is not always the case. Considering processes with relaxation times κ, usually containing terms decaying as $\exp(-\kappa t)$, one can find a PSD depending on ω such as

$$S(\omega) \sim \frac{\kappa}{\kappa^2 + \omega^2} \quad . \tag{1.14}$$

This is called a Lorentzian spectrum and is one type of *colored noise* in analogy to white noise.

1.2.3.1 $1/f^k$ noise

One type of noise is of special interest, since it appears surprisingly often in a wide range of systems. This is a noise with a spectrum shaped as

$$S(\omega) = \frac{C}{\omega^k} \quad , \tag{1.15}$$

with $C =$ constant and $\omega = 2\pi f$. In case of $k \approx 1$ it is referred to as *one-over-f-noise*, *flicker noise* or *pink noise*. Figure 1.10 shows several examples of simulated $1/f^k$ noises for different k. The velocity of Brownian motion, for example, corresponds to such noise with $k = 2$.

A general overview of 1/f noise is provided by [21], [22] and [23]. As mentioned above, examples can be found in a lot of physical and *non-physical* systems [4], as in solids [24]; electronic devices [25] - [27]; magnetic systems [28]; traffic flow [29], [172]; network traffic [30]; neuro systems [31], [176] - [178] and financial data [32].

Especially in solid state physics, many theories and models have been developed and proposed to explain the $1/f$-feature of residence fluctuations [33], [34]. Based on a heuristic theory, $1/f$ noise can be explained as the superposition of Lorentzians (Eq. (1.14)). Each Lorentzian is produced by a relaxation process with a certain waiting time distribution $p(\tau)$. In case of a thermally activated process with $\tau = \tau_0 \exp(E/kT)$ the distribution $p(\tau) \sim 1/\tau$ arises naturally and a noise spectrum close to $1/f$ is obtained. The problem consists now of justifying the distribution, which is often assumed to result from charge trapping. Random-walk models in systems containing traps with broad distributions of activation energies have

[4] A nice bibliography on $1/f^k$ noise can be found at http://linkage.rockefeller.edu/wli/1fnoise/

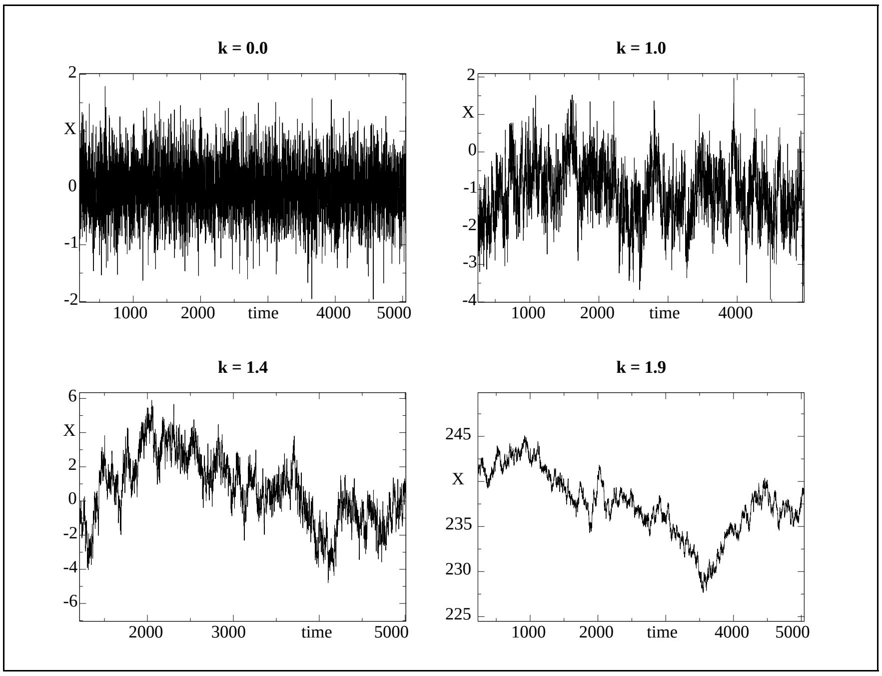

Figure 1.10: Simulated Gaussian noise sequences with different $1/f^k$-shaped spectra.

successfully being used for that investigation [34]. Similar mechanisms might be considered for the explanation of the $1/f$-behavior of other, e.g., non-physical systems.